TRANSITION THEORY OF STRIP BENDING

(PEREKHODNAIA TEORIIA IZGIBA POLOSY)

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1. Introduction. In the ordinary theory of strip bending the following assumptions are made [1].

1) The strip is sufficiently wide, so that it is bent under the conditions of plane strain with no hardening.

2) It is deformed in such a way that its thickness is everywhere the same and the plane edges are transformed into circular cylinders.

3) It has the neutral surface $r = d$ the outer fibers $(r > d)$ being in tension while the inner ones $(r < d)$ are in compression.

4) The condition of yielding

$$\tau_{rr} - \tau_{\theta\theta} = 2k \quad (a < r < d), \qquad \tau_{\theta\theta} - \tau_{rr} = 2k \quad (d < r < b)$$

occurs, a, b being the radii of the cylindrical boundaries in the deformed state.

If we make use of the transition theory of plastic deformation* it can be shown that it is not necessary to assume (4) and that the two regions of yielding - in tension and in compression - follow directly from the equations. It is also found that although in the elastic case

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* Seth, B.R. Elastic-plastic transition in shells and tubes under pressure. Math. Res. Center, Madison, Report, 1962, N 295, 1-18; Elastic-plastic transition in torsion. Report, 1962, n. 302, 1-13.

the strip can acquire various forms of bending, the only possible bending form in the plastic state is circular, provided we assume that the orthogonal nature of the surfaces before the deformation is preserved also after the deformation.

When moment G (per unit width) applied to the edges increases, the elastic deformation becomes nonlinear. In a further increase of G yielding takes place in the plate. The classical model of elastic-plastic body makes use of the yield condition in order to connect the two states. It does not take into account the nonlinear region through which the transition takes place. The theory of finite elastic deformations leads to nonlinear differential equations, the asymptotic solution of which at the bifurcation point represents the transition to the plastic state. This transition due to the change of the stress-strain relations can occur both for tension and compression. Thus two bifurcation points are obtained. It is now sufficient to assume that Poisson's ratio σ is 0.5 and replace the strain tensor by the rate of strain tensor to derive the state of complete plasticity from the transition results. The values of the elasticity coefficients in transition can be expressed in terms of the yield limit in accordance with the results of finite deformation in tension or in shear. We shall show how this theory can be applied to plastic bending of a strip.

2. The components of displacement, strain and stress. Assume that the undeformed planes of the strip $x' = \text{const}$ and $y' = \text{const}$ remain mutually orthogonal after the deformation. Then we may set for the displacements

$$u = x - x' = x - f(\xi), \qquad v = y - y' = y - \varphi(\eta) \tag{2.1}$$

Here $x + iy = z = F(\xi + i\eta) = F(\zeta)$; functions f, φ, F are to be determined.

The components of finite strain referred to the deformed state have the form

$$\begin{aligned} e_{xx} &= \tfrac{1}{2}(1 - f'^2\xi_x^2 - \varphi'^2\eta_x^2) \qquad \left(\xi_x = \frac{\partial\xi}{\partial x}, \ldots\right) \\ e_{yy} &= \tfrac{1}{2}(1 - f'^2\xi_y^2 - \varphi'^2\eta_y^2) \\ e_{xy} &= -\tfrac{1}{2}(f'^2\xi_x\xi_y + \varphi'^2\eta_x\eta_y) \qquad \left(\eta_x = \frac{\partial\eta}{\partial x}, \ldots\right) \end{aligned} \tag{2.2}$$

The stress-strain relations are taken in the form

$$\tau_{ij} = \lambda I_1 + 2\mu e_{ij} \qquad \left(I_1 = 1 - \frac{1}{2}(f'^2 + \varphi'^2)\left|\frac{d\zeta}{dz}\right|\right)^2 \tag{2.3}$$

We have further

$$\tau_{xx} = \lambda I_1 + \mu (1 - f'^2 \xi_x^2 - \varphi'^2 \eta_x^2)$$
$$\tau_{yy} = \lambda I_1 + \mu (1 - f'^2 \xi_y^2 - \varphi'^2 \eta_y^2), \qquad \tau_{xy} = - \mu (f'^2 \xi_x \xi_y + \varphi'^2 \eta_x \eta_y) \tag{2.4}$$

3. Equilibrium equations. The stress equilibrium equations

$$\tau_{ij,j} = 0 \tag{3.1}$$

lead to two nonlinear differential equations

$$\begin{aligned} \frac{\partial}{\partial x}\left[\lambda I_1 - \frac{1}{2}\mu (f'^2 + \varphi'^2) \left|\frac{d\zeta}{dz}\right|^2\right] - \frac{1}{2}\mu \left|\frac{d\zeta}{dz}\right|^2 \frac{\partial}{\partial x}(f'^2 + \varphi'^2) = 0 \\ \frac{\partial}{\partial y}\left[\lambda I_1 - \frac{1}{2}\mu (f'^2 + \varphi'^2) \left|\frac{d\zeta}{dz}\right|^2\right] - \frac{1}{2}\mu \left|\frac{d\zeta}{dz}\right|^2 \frac{\partial}{\partial y}(f'^2 + \varphi'^2) = 0 \end{aligned} \tag{3.2}$$

Setting $F_1 = f'^2 + \varphi'^2$, $F_2 = |d\zeta/dz|^2$ we arrive at the following form of (3.2)

$$\frac{\partial}{\partial x}(F_1 F_2) + (1 - 2\sigma) F_2 \frac{\partial F_1}{\partial x} = 0, \frac{\partial}{\partial y}(F_1 F_2) + (1 - 2\sigma) F_2 \frac{\partial F_1}{\partial y} = 0 \tag{3.3}$$

Their solution has the form

$$F_1^{2(1-\sigma)} F_2 = k_0^2 \tag{3.4}$$

where σ is Poisson's ratio and k_0 is an integration constant.

The functional equation (3.4) can be solved. It can easily be proved that it leads to various forms including circular, hyperbolic, elliptic and hyperelliptic functions. Without going into details of these problems, consider the type of solutions which can occur at the bifurcation points where f' or φ' tends to zero or infinity. It follows from (3.4) that at these points F_2 tends to become a function of ξ or η only. Since $F_2 = |d\zeta/dz|^2$ the corresponding functional equation can be satisfied only when

$$z = \exp(k_0 \zeta) \tag{3.5}$$

where k_0 is a suitable constant. This shows that the plastic deformed state yields a circular cylinder.

4. Transition and plastic stress components. It is easy to derive for the principal stresses $\tau_{\xi\xi}$ and $\tau_{\eta\eta}$ the following expressions:

$$\tau_{\xi\xi} = \lambda I_1 + \mu - \mu F_2 f'^2, \qquad \tau_{\eta\eta} = \lambda I_1 + \mu - \mu F_2 \varphi'^2, \qquad \tau_{\xi\eta} = 0 \tag{4.1}$$

Denoting

$$R = 2 - c - \frac{c}{\mu}\tau_{\xi\xi} = F_2 [(1 - c) F_1 + c f'^2] \qquad \left(c = \frac{1 - 2\sigma}{1 - \sigma}\right) \tag{4.2}$$

we obtain

$$\frac{\partial \ln R}{\partial \xi} = \frac{1}{F_2}\frac{\partial F_2}{\partial \xi} + \frac{\partial F_1 / \partial \xi - c\, \partial \varphi'^2 / \partial \xi}{(1-c) F_1 + c f'^2} \tag{4.3}$$

or making use of (3.4)

$$\frac{\partial \ln R}{\partial \xi} = \frac{\partial \ln F_2}{\partial \xi} - \frac{(1 - c/2) F_1\, \partial \ln F_2 / \partial \xi - c\, \partial \varphi'^2 / \partial \xi}{(1-c) F_1 + c f'^2} \tag{4.4}$$

An analogous equation is obtained if we differentiate (4.2) with respect to η.

As $f' \to \infty$, which corresponds to an infinite tension, we obtain from (4.4)

$$\frac{\partial \ln R}{\partial \xi} = \frac{c}{2}\frac{\partial \ln F_2}{\partial \xi}, \qquad \frac{\partial \ln R}{\partial \eta} = \frac{c}{2}\frac{\partial \ln F_2}{\partial \eta} \tag{4.5}$$

(The second equation is derived in an analogous way.) These equations prove that in the transition

$$R = K_1 F_2^{c/2} \tag{4.6}$$

where K_1 is the integration constant.

Similarly, when $f' \to 0$, which corresponds to an infinite compression, we obtain for the transition value of R the formula

$$R = K_2\, [F_2^{1/2}\, \varphi'^2]^{-c/(1-c)} \tag{4.7}$$

Taking into account (4.1) and (4.2) we arrive at the transition values of $\tau_{\xi\xi}$ and $\tau_{\eta\eta}$.

As it was mentioned in Section 3, quantities ξ and η are polar coordinates so that $z = \exp \zeta$ and $F_2 = |d\zeta/dz|^2 = 1/r^2$. Consequently, from (4.6) and (4.7) we obtain

$$\begin{aligned} \tau_{rr} &= \frac{\mu(2-c)}{c}\left[1 - \left(\frac{b}{r}\right)^{c}\right] \quad \text{in the tension region} \\ \tau_{rr} &= \frac{\mu(2-c)}{c}\left[1 - \left(\frac{r}{a}\right)^{c/(1-c)}\right] \quad \text{in the compression region} \end{aligned} \tag{4.8}$$

where a and b are the interior and exterior radii of the deformed strip.

It follows from the results for simple shear under the conditions of finite deformation [2], that in transition $\mu \to k$ the latter being the yield limit in shear. Passing to the limit $c \to 0$ (i.e. $\sigma \to 1/2$) we obtain from (4.8) for the plastic radial stress

$$\tau_{rr} = -2k \ln \frac{b}{r} \quad (d < r < b), \qquad \tau_{rr} = -2k \ln \frac{r}{a} \quad (a < r < d) \tag{4.9}$$

the boundary between the regions of tension and compression being defined by the relation

$$d^2 = ab \tag{4.10}$$

From (4.1) we obtain

$$\tau_{\theta\theta} = 2k\left(1 - \ln\frac{b}{r}\right) \quad (d < r < b), \qquad \tau_{\theta\theta} = -2k\left(1 + \ln\frac{r}{a}\right) \quad (a < r < d) \tag{4.11}$$

This results in the yield condition mentioned in Section 1.

5. The nature of the deformation. As soon as the flow begins, instead of the strain components the rate of strain has to be employed, and the transition state has to be taken as initial. The definition of the rate of strain contains only instantaneous and infinitely close to it configurations. Therefore we may take

$$2e_{ij} = u_{i,j} + u_{j,i} \tag{5.1}$$

instead of the components of the finite deformation used in Section 2. Thus we obtain

$$\dot{e}_{ij} = \lambda_1\left(\frac{3}{2}\tau_{ij} - \frac{1}{2}\tau_{ii}\right) \qquad (\theta_{ii} = 0) \tag{5.2}$$

Here point denotes differentiation with respect to a conveniently selected parameter of the deformation process. This relation can be obtained from the following relation used in Section 2

$$e_{ij} = E^{-1}\left[(1 + \sigma)\tau_{ij} - \sigma\tau_{ii}\right] \tag{5.3}$$

setting $\sigma = 1/2$ and replacing E^{-1} by λ and e_{ij} by $\dot{e}_{ij}$. It is also assumed that the elastic deformation is negligibly small.

For the plastic state ($\sigma \to 1/2$, $c \to 0$) we obtain from (4.2)

$$f'^2 = 2 - \frac{A^2}{r^2}, \qquad A^2 = d^2 = ab \tag{5.4}$$

Let α be the angle of bending per unit length and $\ln\alpha$ the parameter of yielding. Then

$$\frac{\partial e_{rr}}{\partial \ln\alpha} = \frac{1}{2}(1 - f'^2) = \frac{1}{2}\left(\frac{d^2}{r^2} - 1\right), \qquad \frac{\partial e_{\theta\theta}}{\partial \ln\alpha} = \frac{1}{2}\left(1 - \frac{d^2}{r^2}\right) \tag{5.5}$$

Consequently

$$\delta u = -\frac{1}{2}\left(r + \frac{d^2}{r}\right)\frac{\delta\alpha}{\alpha}, \qquad \delta v = r\theta\,\frac{\delta\alpha}{\alpha} \tag{5.6}$$

The components of the radial and transverse deformations (5.6) are identical to those derived by Hill [1]. The remaining analysis is the same as Hill's.

BIBLIOGRAPHY

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2. Seth, B.R., Nonlinear continuum mechanics. Presidential address. Section of Mathematics Proc. 42nd Ind. Sci. Congr., II, p.8, 1955.

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